International Journal of Theoretical Physics, Vol. 40, No. 11, November 2001 (© 2001)

Open Universe Solution in the Vacuum-Induced-Gravity Theory

Albert Nzabonimana[1] and Liu Liao[1,2]

We show that there is an open universe solution in the vacuum-induced-gravity theory ($k = 0$). The scale factor $a(t)$ obeys a power law $a(t) \sim t^{\alpha}$, where α is a real number. Under some constraint conditions, this solution is similar to the Einstein–De Sitter universe solution of Einstein's theory.

A theory of gravity incorporating the concept of spontaneous symmetry breaking is proposed by Zee (1979). It is suggested that the same symmetry breaking mechanism is responsible for breaking a unified gauge theory into strong, weak, and electromagnetic interactions.

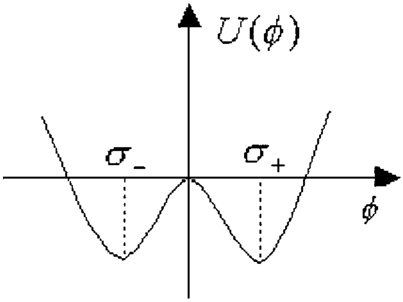
Einstein's theory of gravity and Fermi's theory of weak interaction have a feature in common. In contrast to electrodynamics and modern theory of strong interaction, their coupling constants have dimension of $M^{-2} \equiv (\text{mass})^{-2}$. They are notably small, with Fermi's coupling constant $G_{\mathrm{F}} = (300\, m_{\mathrm{N}})^{-2}$ and Newton's coupling constant $G_{\mathrm{N}} = (10^{19}\, m_{\mathrm{N}})^{-2}$. It has been suggested that the smallness of G_{F} is due to the massiveness of the intermediate boson.

Central to the unification scheme is the concept of spontaneous symmetry breaking, i.e. the Higgs potentials have an asymmetric vacuum expectation value σ, thus generating the mass of intermediate boson, so that

$$G_{\mathrm{F}} \sim \sigma^{-2}$$

[1] Department of Physics and Institute of Theoretical Physics, Beijing Normal University, Beijing 100875, People's Republic of China.
[2] Correspondence should be addressed at Department of Physics and Institute of Theoretical Physics, Beijing Normal University, Beijing 100875, People's Republic of China.

0020-7748/01/1100-2085$19.50/0



The concept of spontaneous symmetry breaking has proved to be extraordinary fruitful in many areas of physics, and it is worthwhile to try to incorporate it into gravitation. Motivated by these considerations, Zee suggested that Einstein's action for gravity should be modified to

$$S = \int d^4x(-g)^{1/2}\left\{\frac{1}{2}\varepsilon\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + L_{\rm w}\right\} \tag{1}$$

Here ϕ represents the Higgs field and ε denotes a dimensionless coupling constant that we could take to be order of $0 < \varepsilon \leq 1$. $L_{\rm w}$ is the Lagrangian for the rest of the world: we must distinguish two physically different cases that depend on whether or not $L_{\rm w}$ includes ϕ, that is to say whether or not ϕ interacts with matter fields directly. For the moment, we will assume that $L_{\rm w}$ does not include ϕ and $V(\phi) = \frac{1}{8}\lambda(\phi^2 - \nu^2)^2$ is just the Higgs potential, which is minimized at $\phi = \pm\sigma$.

Because of the fact that we assume the universe is homogeneous and isotropic, the scale factor of the universe and the Higgs field both depend only on time coordinate t, i.e.

$$a \equiv a(t), \qquad \phi \equiv \phi(t). \tag{2}$$

From Robertson–Walker metric

$$ds^2 = dt^2 - a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right\}, \tag{3}$$

we get the scalar curvature

$$R = R^u_\mu = -6\{\ddot{a}a^{-1} + (\dot{a}a^{-1})^2 + \kappa a^{-2}\}.$$

Thus, the Lagrangian in induced gravity becomes

$$\begin{aligned} L &= (-g)^{1/2}\left\{\frac{1}{2}\varepsilon\phi^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + L_{\rm w}\right\} \\ &= (-g)^{1/2}\left\{\frac{1}{2}\varepsilon\phi^2 R - \frac{1}{2}g^{tt}(\partial_t\phi)^2 - V(\phi) + L_{\rm w}\right\} \end{aligned}$$

$$= a^3 r^2 (1 - \kappa r^2)^{-1/2} \sin\theta \left\{ -3\varepsilon\phi^2 [\ddot{a}a^{-1} + \dot{a}^2 a^{-2} + k a^{-2}] + \left(-\frac{1}{2}\right)(\dot{\phi})^2 - \frac{1}{8}\lambda(\phi^2 - \sigma^2)^2 \right\},$$

where we assume $L_{\mathrm{w}} = 0$, $V(\phi) = \frac{1}{8}\lambda(\phi^2 - \sigma^2)^2$, and λ is a very small constant $\lambda \ll 1$.

$$\dot{a} = \frac{d(a(t))}{dt}, \qquad \ddot{a} = \frac{d^2(a(t))}{dt^2}$$

$$\dot{\phi} = \frac{d(\phi(t))}{dt}, \qquad \ddot{\phi} = \frac{d^2(\phi(t))}{dt^2}.$$

By variation with respect to a and ϕ, we obtain a system of two equations, i.e. $a(t)$ equation

$$\frac{\partial L}{\partial a} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{a}}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial L}{\partial \ddot{a}}\right) = 0 \tag{4}$$

and $\phi(t)$ equation

$$\frac{\partial L}{\partial \phi} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0,\ L(a, \phi, \dot{a}, \dot{\phi}, \ddot{a}). \tag{5}$$

That is,

$$\left\{ \varepsilon\phi^2[-2\ddot{a}\,a - \dot{a}^2 - k] - 2\varepsilon a^2 \phi\,\ddot{\phi} - 4\varepsilon a\,\dot{a}\,\phi\,\dot{\phi} - \frac{1}{2}a^2(\dot{\phi})^2(1 + 4\varepsilon) - \frac{1}{8}\lambda a^2(\phi^2 - \sigma^2)^2 \right\} = 0$$

$$\sim a(t) \text{ equation,} \tag{6}$$

and

$$-6\varepsilon\phi[\ddot{a}\,a + \dot{a}^2 + k] - \frac{1}{2}\lambda a^2(\phi^2 - \sigma^2)\phi + 3a\,\dot{a}\,\dot{\phi} + a^2\ddot{\phi} = 0$$

$$\sim \phi(t) \text{ equation.} \tag{7}$$

Because of the fact that we assume $\phi(t) \sim \sigma$ and the smallness of constant λ, we could neglect the terms $-\frac{1}{8}\lambda a^2(\phi^2 - \sigma^2)^2 \sim 0$ and $-\frac{1}{2}\lambda a^2(\phi^2 - \sigma^2) \sim 0$ in these system of equations.

So in the neighborhood of $\phi = \sigma$, the same system of equations approximately become as shown here:

$$\varepsilon\left\{ -2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} \right\} - 2\varepsilon\frac{\ddot{\phi}}{\phi} - 4\varepsilon\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} - \frac{1}{2}(1 + 4\varepsilon)\left(\frac{\dot{\phi}}{\phi}\right)^2 = 0$$

$$\sim a(t) \text{ equation,} \tag{8}$$

and

$$-6\varepsilon\left\{\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2+\frac{k}{a^2}\right\}+3\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi}+\frac{\ddot{\phi}}{\phi}=0$$
$$\sim\phi(t) \text{ equation.} \tag{9}$$

For $k=0$, the system of equations $a(t)$, and $\phi(t)$ changes to the following form:

$$\varepsilon\left\{-2\frac{\ddot{a}}{a}-\left(\frac{\dot{a}}{a}\right)^2\right\}-2\varepsilon\frac{\ddot{\phi}}{\phi}-4\varepsilon\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi}-\frac{1}{2}(1+4\varepsilon)\left(\frac{\dot{\phi}}{\phi}\right)^2=0 \tag{10}$$

$$-6\varepsilon\left\{\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right\}+3\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi}+\frac{\ddot{\phi}}{\phi}=0. \tag{11}$$

Let

$$A(t)=\frac{\dot{a}(t)}{a(t)} \quad \text{or} \quad \frac{\ddot{a}(t)}{a(t)}=\dot{A}(t)+A^2(t),$$
$$\Theta(t)=\frac{\dot{\phi}(t)}{\phi(t)} \quad \text{or} \quad \frac{\ddot{\phi}(t)}{\phi(t)}=\dot{\Theta}(t)+\Theta^2(t). \tag{12}$$

Then, these system of equations become

$$\varepsilon\{4\dot{A}+6A^2\}+8\varepsilon A\Theta+4\varepsilon\dot{\Theta}+(1+8\varepsilon)\Theta^2=0 \tag{13}$$
$$-6\varepsilon\{\dot{A}+2A^2\}+3A\Theta+\dot{\Theta}+\Theta^2=0. \tag{14}$$

Now, we try to eliminate the cross term $A\Theta$, by taking $3\times a(t)$ equation $-8\varepsilon\times\Phi(t)$ equation: therefore the following nonlinear differential equation results:

$$D(\dot{A},A^2,\dot{\Theta},\Theta^2)$$
$$\equiv(48\varepsilon^2+12\varepsilon)\dot{A}+(96\varepsilon^2+18\varepsilon)A^2+4\varepsilon\dot{\Theta}+(16\varepsilon+3)\Theta^2=0.$$

Let $D^*=\frac{D}{48\varepsilon^2+12\varepsilon}$.
It implies that

$$D^*(\dot{A},A^2,\dot{\Theta},\Theta^2)=\dot{A}+\left(\frac{96\varepsilon^2+18\varepsilon}{48\varepsilon^2+12\varepsilon}\right)A^2+\frac{4\varepsilon}{48\varepsilon^2+12\varepsilon}\dot{\Theta}$$
$$+\left(\frac{3+16\varepsilon}{48\varepsilon^2+12\varepsilon}\right)\Theta^2=0.$$

From $D^*(\dot{A}, A^2, \dot{\Theta}, \Theta^2) = 0$, we have

$$\dot{A} + \left(\frac{96\varepsilon^2 + 18\varepsilon}{48\varepsilon^2 + 12\varepsilon}\right)A^2 = -\frac{4\varepsilon}{48\varepsilon^2 + 12\varepsilon}\dot{\Theta} - \left(\frac{3 + 16\varepsilon}{48\varepsilon^2 + 12\varepsilon}\right)\Theta^2$$
$$= c_1 \text{ (constant)} \tag{15}$$

$$\text{or} \quad \dot{A} + m_1 A^2 = c_1, \tag{16}$$

$$\dot{\Theta} + m_2 \Theta^2 = c_2, \tag{17}$$

where

$$m_1 = \frac{96\varepsilon^2 + 18\varepsilon}{48\varepsilon^2 + 12\varepsilon} > 0, \qquad m_2 = \frac{3 + 16\varepsilon}{4\varepsilon} > 0$$
$$c_2 = -\frac{48\varepsilon^2 + 12\varepsilon}{4\varepsilon}c_1, \quad 0 < \varepsilon \le 1.$$

Let us now prove that $c_1 = 0$: Both Eqs. (16) and (17) are Ricatti differential equations. If we suppose $c_1 > 0$, then $c_2 = -(48\varepsilon^2 + 12\varepsilon)c_1/4\varepsilon < 0$: the solution of Eqs. (16) and (17) will become respectively

$$A(t) = \frac{A_0\sqrt{m_1 c_1} + c_1 th(\sqrt{m_1 c_1}(t - t_0))}{\sqrt{m_1 c_1} + m_1 A_0 th(\sqrt{m_1 c_1}(t - t_0))}, \quad m_1 c_1 > 0 \tag{18}$$

and

$$\Theta(t) = \frac{\Theta_0\sqrt{-m_2 c_2} + c_2 tg(\sqrt{-m_2 c_2}(t - t_0))}{\sqrt{-m_2 c_2} + m_2 \Theta_0 tg(\sqrt{-m_2 c_2}(t - t_0))}, \quad m_2 c_2 < 0. \tag{19}$$

But now $A(t)$ and $\Theta(t)$ do not satisfy Eqs. (13) and (14), i.e. they are not the solutions of Eqs. (13) and (14). With the same procedure, in the case of $c_1 < 0$, we get similarly $A(t)$ and $\Theta(t)$ that are again not the solution of Eqs. (13) and (14). So, the only case to discuss is $c_1 = 0$, i.e.

$$\dot{A} + m_1 A^2 = 0 \tag{20}$$

and

$$\dot{\Theta} + m_2 \Theta^2 = 0. \tag{21}$$

The solutions of Eqs. (20) and (21) for $k = 0$ are

$$a(t) = \exp\left[\int A(t)\,dt\right] = \left|1 + \left(\frac{16\varepsilon + 3}{8\varepsilon + 2}\right)A_0(t - t_0)\right|^{\left(\frac{8\varepsilon+2}{16\varepsilon+3}\right)} \tag{22}$$

and

$$\phi(t) = \exp\left[\int \Theta(t)\,dt\right] = \left|1 + \left(\frac{16\varepsilon + 3}{4\varepsilon}\right)\Theta_0(t - t_0)\right|^{\left(\frac{4\varepsilon}{16\varepsilon+3}\right)} \tag{23}$$

respectively, with Θ_0 and A_0, which are constants and depend only on initial conditions. It can be proved that the Solutions (22) and (23) are really the solutions of Eqs. (13) and (14).

In sum, there exist an initial curvature singularity point of the open universe ($k = 0$) in the induced-gravity theory. At $t = \{t_0 - (\frac{8\varepsilon+2}{16\varepsilon+3})\frac{1}{A_0}\}$, the distance between all points of space was zero and the curvature of space–time were infinite $R = R^{\mu}_{\mu} \sim \infty$, $R_{\mu\nu}R^{\mu\nu} \sim \infty$, and $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} \sim \infty$.

This universe is in expansion forever and the expansion of the universe for $k = 0$ obeys a power law $a(t) \sim t^{\alpha}(t \to \infty)$ with $\alpha = \frac{8\varepsilon+2}{16\varepsilon+3} > 0$.

When $\varepsilon \to 0$, $t \to \infty$, the solution of an open universe ($k = 0$) in the induced gravity is similarly to Einstein–De Sitter universe $a(t) \sim t^{2/3}$, dominated by pressureless matter ($\rho \neq 0$, $p = 0$) (See, Liu, 1987).

REFERENCES

Zee, A. (1979). Broken-symmetric theory of gravity. *Physical Review Letters* **42**, 417.
Wald, R. M. (1982). *General Relativity* **92**, 107.
Liu, L. (1987). *Theory of General Relativity*, High Education Press, pp. 447.